Indian J. Phys. **43**, 92–103 (1969)

Wave equation of charged particle of spin 1 — the Kemmer equation

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(Received 18 December 1968)

The Kemmer relativistic wave equation of a particle of spin one or zero and particularly the commutation relations of Duffin-kemmer β-matrices have been obtained by the generalization of the corresponding non-relativistic wave equation. This covariant formulation is achieved by making a Lorenz-transformation to a reference frame in which the velocity of the particle is relatively high from a reference frame in which the velocity is small and also by generalizing the concept of spin and spin space.

Feshback & Villars (1958) derived the Dirac equation by the generalization of the non-relativistic Pauli equation in an unique manner. They started with the particle in the reference frame in which the Pauli theory applies and the wave function, ϕ, has two components corresponding to two orientations of the spin. The relativistic description is then obtained by making a Lorentz transformation to a reference frame in which the velocity of the particle is relatively high. For this purpose they generalized the concept of spin and spin space, and sought for covariant equation in that space. The purpose of this paper is to show that by following their procedure a similar derivation can also be made of the Kemmer (1939) wave equation of a particle of spin one or zero, particularly in obtaining the commutation relations of Duffin-Kemmer (Duffin, 1939) β-matrices.

The non-relativistic Hamiltonian of a particle of charge e and the intrinsic magnetic moment μ in a given electromagnetic field defined by the vector potential $\boldsymbol{A}$ and the scalar potential A_o is given by

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 + eA_o - (\mu.\boldsymbol{h}).$$

The wave equation is then

$$i\hbar\frac{\partial\phi}{\partial t} = \frac{1}{2m}\left(-i\hbar\, grad - \frac{e}{c}\boldsymbol{A}\right)^2\phi + eA_o\,\phi - (\mu.\boldsymbol{h})\,\phi, \tag{1}$$

where

$$\mu = \frac{e\hbar}{mc}\boldsymbol{S}, \tag{2}$$

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$\boldsymbol{S}$ being the infinitesimal rotation operator in the spin space. We assume the commutation rules for the components of $\boldsymbol{S}$ as

$$S_iS_j-S_jS_i=iS_k, \tag{3}$$

where i, j, k are cyclic permutations of 1, 2, 3, which one expects for components of any angular momentum. When expressed in covariant form, (3) becomes

$$S_iS_j-S_jS_i=i\epsilon_{ijk}S_k. \tag{4}$$

Further we assume that

$$S_i^3=S_i,\ (i=1, 2, 3), \tag{5}$$

so that the eigenvalues of S_i are $-1, 0, 1$, which means that the particle has spin 1. The wave function ϕ in (1) then has three components corresponding to the three independent orientations of the spin.

In the irreducible representation in which S_3 is diagonal (3) and (5) are satisfied by (Powell & Crasemann, 1963)

$$S_1=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\ S_2=\frac{i}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},\ S_3=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix} \tag{6}$$

One can show, as a consequence of (4) and (5), that S_i $(i=1, 2, 3)$ satisfy the commutation relations

$$S_iS_jS_k+S_kS_jS_i=S_i\delta_{jk}+S_k\delta_{ji} \tag{7}$$

which when written in details are equivalent to

$$S_i^3=S_i \tag{8a}$$

$$S_i^2S_j+S_jS_i^2=S_j,\ (i\neq j) \tag{8b}$$

$$S_iS_jS_i=0,\ (i\neq j) \tag{8c}$$

$$S_iS_jS_k+S_kS_jS_i=0,\ (i\neq j\neq k). \tag{8d}$$

To see this, multiply (3) on the left by S_i and on the right also by S_i and on subtraction one obtains

$$S_i^2S_j+S_jS_i^2-2S_iS_jS_i=-i(S_kS_i-S_iS_k)=S_j \tag{9}$$

Multiply this equation on both sides by S_i and using (5), one gets

$$S_i S_j S_i = 2 S_i^2 S_j S_i^2$$

Again multiplying on both sides by S_i

$$S_i^2 S_j S_i^2 = 2 S_i S_j S_i.$$

Thus one derives (8c) and hence (8b) from (9). To obtain (8d) multiply (3) on both sides by S_i and S_j, again reverse this order of multiplication and add deriving

$$S_i^2 S_j^2 - S_j^2 S_i^2 = i(S_i S_k S_j + S_j S_k S_i), \ (i \neq j \neq k),$$

where (8c) has been used. Now by repeated application of (8b) one proves that

$$S_i^2 S_j^2 = S_j^2 S_i^2$$

whence (8d) is verified.

The invariance of the commutation relation (7) or of (4) and (5) under an orthogonal transformation

$$S_k' = a_{ki} S_i, \ \ a_{ki} a_{kj} = \delta_{ij}, \ (k, i, j = 1, 2, 3), \tag{10}$$

a_{ik} being the transformation matrix relating the two coordinate systems, is evident. Clearly the transformation (10) changes the representation of the $\boldsymbol{S}$ matrices.

The relativistic description is now obtained by making a Lorentz transformation to the reference system in which the velocity is large. For this purpose we identify that $\boldsymbol{S}$ is the space-part of an antisymmetrical tensor $\Sigma_{\mu\nu}$ – that is the rotation operator $\boldsymbol{S}$, whose components have the properties (8a)–(8d), is a pseudovector constructed from the spatial components of $\Sigma_{\mu\nu}$. Thus

$$\Sigma_{\mu\nu} = -\Sigma_{\nu\mu}, \ \ \boldsymbol{S} = (\Sigma_{23}, \Sigma_{31}, \Sigma_{12}). \tag{11}$$

The space-time part we denote by the vector operator

$$\boldsymbol{T} = (\Sigma_{14}, \Sigma_{24}, \Sigma_{34}) \tag{12}$$

We must regard $\boldsymbol{T}$ as unknown for the present. Our first problem is to find the relations between $\boldsymbol{T}$ and $\boldsymbol{S}$ and also the commutation rules of $\boldsymbol{T}$. Under a Lorentz transformation, $\Sigma_{\mu\nu}$ is transformed to

$$\Sigma_{\mu\nu}'=a_{\mu\rho}\,a_{\nu\sigma}\Sigma_{\rho\sigma}.$$

For instance, under a Lorentz transformation for a motion of the primed reference system relative to an unprimed one, in which the velocity of the particle is small, with velocity v in the x_1-direction, we obtain

$$S_1'=S_1,\ S_2'=\xi\left(S_2+i\frac{v}{c}T_3\right),\ S_3'=\xi\left(S_3-i\frac{v}{c}T_2\right), \tag{13a}$$

and

$$T_1'=T_1,\ T_2'=\xi\left(T_2+i\frac{v}{c}S_3\right),\ T_3'=\xi\left(T_3-i\frac{v}{c}S_2\right) \tag{13b}$$

where $\xi=\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$. Similar relations follow for motions in the x_2- and x_3-directions, which can be obtained from (13a) and (13b) by cyclic permutations of 1, 2, 3.

The relations (8a) — (8d) should be valid for any inertial system of reference, and hence in the primed system also. Then from $S_2'^3=S_2'$ and $S_3'^3=S_3'$, we have

$$\left.\begin{aligned} S_2^2T_3+T_3S_2^2+S_2T_3S_2&=T_3\\ T_3^2S_2+S_2T_3^2+T_3S_2T_3&=S_2,\\ T_3^3&=T_3\end{aligned}\right\} \tag{14}$$

with similar relations obtained by the interchange of 2 and 3.

Similar relations are obtained for motions along x_2- and x_3-directions, by cyclic permutations of 1,2,3.

To see (14) we substitute S_2' from (13a) in $S_2'^3=S_2'$ and get

$$S_2^3+i\frac{v}{c}\left(S_2^2T_3+T_3S_2^2+S_2T_3S_2\right)-\frac{v^2}{c^2}\left(T_3^2S_2+S_2T_3^2+T_3S_2T_3\right)$$

$$-i\frac{v^3}{c^3}T_3^3=\left(1-\frac{v^2}{c^2}\right)\left(S_2+i\frac{v}{c}T_3\right)$$

Equating the co-efficients of each power of $\frac{v}{c}$ on both sides of this equation we obtain the set (14). Similarly from $S_3'^3 = S_3'$.

In a similar way, $S'^2_i S'_j + S'_j S'^2_i = S'_j$, $(i \neq j)$, gives for $i, j = (2,1)$, (3,1), (1,2), (1,3), (2,3), (3,2)

$$\left.\begin{array}{c} T_3^2 S_1 + S_1 T_3^2 = S_1, \\ (S_1 T_3 S_2 + S_2 T_3 S_1) + (S_1 S_2 T_3 + T_3 S_2 S_1) = 0, \\ \text{(with the interchange of 2 and 3)} \end{array}\right\} \qquad \text{...(15)}$$

$$S_1^2 T_3 + T_3 S_1^2 = T_3, \ S_1^2 T_2 + T_2 S_1^2 = T_2, \qquad \text{...(16)}$$

$$\left.\begin{array}{c} (S_2 T_3 S_3 + S_3 T_3 S_2) + (T_3 S_2 S_3 + S_3 S_2 T_3) - (S_2^2 T_2 + T_2 S_2^2) = -T_2, \\ (T_3 S_2 T_2 + T_2 S_2 T_3) + (S_2 T_3 T_2 + T_2 T_3 S_2) - (T_3^2 S_3 + S_3 T_3^2) = -S_3, \\ T_3^2 T_2 + T_2 T_3^2 = T_2, \\ \text{(with the interchange of 2 and 3)} \end{array}\right\} \qquad (17)$$

Further $S_i' S_j' S_i' = 0$, $(i \neq j)$, contributes for $i, j = (2,1)$, (3,1), (1,2), (1,3) (2,3), (3,2)

$$\left.\begin{array}{c} S_2 S_1 T_3 + T_3 S_1 S_2 = 0, \\ T_3 S_1 T_3 = 0, \\ \text{(with the interchange of 2 and 3)} \end{array}\right\} \qquad \text{...(18)}$$

$$S_1 T_3 S_1 = 0, \ S_1 T_2 S_1 = 0, \qquad \text{...(19)}$$

$$\left.\begin{array}{c} S_2 T_2 S_2 = T_3 S_3 S_2 + S_2 S_3 T_3, \\ T_3 S_3 T_3 = S_2 T_2 T_3 + T_3 T_2 S_2, \\ T_3 T_2 T_3 = 0, \\ \text{(with the interchange of 2 and 3)} \end{array}\right\} \qquad \text{...(20)}$$

Finally $S_i' S_j' S_k' + S_k' S_j' S_i' = 0$, $(i \neq j \neq k)$, gives for $i, j, k = (1,2,3)$, (1,3,2) and (2,1,3)

$$\left.\begin{array}{c} (S_1 T_3 S_3 + S_3 T_3 S_1) - (S_1 S_2 T_2 + T_2 S_2 S_1) = 0, \\ S_1 T_3 T_2 + T_2 T_3 S_1 = 0, \\ \text{(with the interchange of 2 and 3)} \end{array}\right\} \qquad \text{...(21)}$$

$$\left.\begin{array}{c} (T_3 S_1 S_3 + S_3 S_1 T_3) - (S_2 S_1 T_2 + T_2 S_1 S_2) = 0, \\ T_3 S_1 S_2 + T_2 S_1 T_3 = 0, \end{array}\right\} \qquad \text{...(22)}$$

The relations similar to (15)–(22) are obtained for transformations in the x_2-x_4 and x_3-x_4 plane by the cyclic permutations of 1,2,3.

Collecting the results one has the following relations between T_k :

$$\left.\begin{aligned} T_i^2T_j + T_jT_i^2 &= T_j,\ (i\neq j), \\ T_iT_jT_i &= 0,\ (i\neq j), \\ T_i^3 &= T_i, \end{aligned}\right\} \qquad ...(23)$$

and the relations between S_k and T_k are given by

$$S_i^2T_j + T_jS_i^2 = T_j,\ (i\neq j), \qquad ...(24a)$$

$$T_i^2S_j + S_jT_i^2 = S_j,\ (i\neq j), \qquad ...(24b)$$

$$S_iT_jS_i = 0,\ (i\neq j), \qquad ...(25a)$$

$$T_iS_jT_i = 0,\ (i\neq j), \qquad ...(25b)$$

$$S_iT_iS_i = T_jS_jS_i + S_iS_jT_j,\ (i\neq j), \qquad ...(26a)$$

$$T_iS_iT_i = S_jT_jT_i + T_iT_jS_j,\ (i\neq j), \qquad ...(26b)$$

$$S_iS_jT_k + T_kS_jS_i = 0,\ (i\neq j\neq k), \qquad ...(27a)$$

$$T_iT_jS_k + S_kT_jT_i = 0,\ (i\neq j\neq k), \qquad ...(27b)$$

$$T_iS_jT_k + T_kS_jT_i = 0,\ (i\neq j\neq k), \qquad ...(27c)$$

$$S_iT_jS_k + S_kT_jS_i = 0,\ (i\neq j\neq k), \qquad ...(27d)$$

$$S_iT_jS_j+S_jT_jS_i+T_jS_iS_j+S_jS_iT_j-(S_i^2T_i+T_iS_i^2)= -T_i,\ (i\neq j), \qquad ...(28a)$$

$$T_iS_jT_j+T_jS_jT_i+S_jT_iT_j+T_jT_iS_j-(T_i^2S_i+S_iT_i^2)= -S_i,\ (i\neq j), \qquad ...(28b)$$

$$(S_iT_jS_j+S_jT_jS_i)-(S_iS_kT_k+T_kS_kS_i)=0,\ (i\neq j\neq k), \qquad ..\ (29a)$$

$$(T_jS_iS_j+S_jS_iT_j)-(S_kS_iT_k+T_kS_iS_k)=0,\ (i\neq j\neq k), \qquad ...(29b)$$

The relation (27d) has been obtained from the second set of (15) and (27a). The first set of (14) is satisfied by (24a) and (25a), and the second set by (24b) and (25b). It should be noticed that the relations between three T's with different indices are absent in (23). This is caused by our restriction to Lorentz transformation for motion in the direction of coordinate axes only, in which case only two of the components of $\boldsymbol{S}$ are transformed and one remains unchanged.

For further discussion we use the following result :—Any vector $\boldsymbol{V}$ in spin-space satisfies the commutations relations (Powell & Crasemann, 1963)

$$[S_i, V_i]=0, \ [S_i, V_j]= -[S_j, V_i]=iV_k, \ (i, j, k=\text{cyclic})$$

an equation, which in terms of the Σ_{ik} $(i, k = 1,2,3)$ may be written as

$$[V_i, \Sigma_{jk}] = - i(\delta_{ij}V_k - \delta_{ik}V_j) \qquad ...(30)$$

Indentifying $\boldsymbol{T}$ with $\boldsymbol{V}$, we have the relations

$$[S_i, T_i] = 0, \ [S_i, T_j] = - [S_j, T_i] = iT_k, \ (i, j, k=\text{cylic}) \qquad .. (31)$$

The simplest solution of this equation for S_k in terms of T_j may be expressed as a product of two T-matrices. The detail working of the method, that we have followed here, is the same as has been used later for the derivation of the commutation relations for β-matrices. To avoid repetition of calculations we give here only the main results. Since $S_k=\Sigma_{ij}$ is an antisymmetrical tensor of rank two, we obtain

$$i S_k=(T_iT_j-T_jT_i), \ (i, j, k=\text{cyclic}) \qquad ...(32)$$

with the three commutation relations (23) for T_i and in addition

$$T_i T_jT_k+T_kT_jT_i=0, \ (i\neq j\neq k), \qquad ...(33)$$

The relations (23) and (33) between T's can then be expressed in co-variant form as

$$T_iT_jT_k+T_kT_jT_i=T_i\delta_{jk}+T_k\delta_{ji} \qquad .. (34)$$

With the use of $[S_i, T_i]=0$, (26a) and (26b), the relations (28a) and (28b) can be simplified to

$$T_jS_iS_j+S_jS_iT_j-S_iT_jS_i= - T_i, \ (i\neq j), \qquad ...(28a')$$

$$S_jT_iT_j+T_jT_iS_j-T_iS_iT_i = - S_i, \ (i\neq j) \qquad ...(28b')$$

and (29a) and (29b) are identically satisfied.

The relations (24a)—(27d), (28a') and (28b') give all possible relations between two S's and one T, and between two T's and one S. By a simple but tedious calculations it can be shown that they are all identically satisfied by (32), which has been obtained from (31). Thus these relations together with (34) are consistent with (31) and can also be obtained directly from it (See the Appendix).

We have thus obtained the properties of the rotation operator $\Sigma_{\mu\nu}$ in spin space. One should notice that the relations (7) and (34) are the commutation relations for the Duffin-Kemmer β-matrices. But there is one difference, the fundamental Duffin-Kemmer matrices are four, whereas we have here only three S-matrices and three T-matrices. We shall now prove that there exists a four-vector β_μ in spin space having the algebraic properties.

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu \beta_\mu \delta_{\nu\lambda} + \beta_\lambda \delta_{\nu\mu} \qquad \ldots(35)$$

or

$$\beta_\mu \beta_\nu - \beta_\nu \beta_\mu = i\Sigma_{\mu\nu}, \quad \beta_\mu{}^3 = \beta_\mu \qquad \ldots(36)$$

For this purpose, a general infinitesimal Lorentz transformation is performed on β_μ, such that

$$\exp\left(\tfrac{1}{2} i \Sigma_{\alpha\beta}\omega_{\alpha\beta}\right)\beta_\mu \exp\left(-\tfrac{1}{2}\, i\, \Sigma_{\alpha\beta}\omega_{\alpha\beta}\right) = a_{\mu\nu}(\omega_{\alpha\beta})\, \beta_\nu \qquad \ldots(37)$$

where $a_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu}$, $\omega_{\mu\nu} = -\omega_{\nu\mu}$, are the co-officients in Lorentz transformation and the Einstein summation rule has been followed. The commutation relations between $\Sigma_{\alpha\beta}$ and β_μ are then obtained from (37) and the result will be a covariant generalization of (30) which is given by

$$[\beta_\mu, \Sigma_{\lambda\nu}] = -i[\delta_{\mu\lambda}\beta_\nu - \delta_{\mu\nu}\beta_\lambda], \quad (\mu,\nu,\lambda = 1, 2, 3, 4.) \qquad \ldots(38)$$

If we put Σ_{23}, Σ_{31}, $\Sigma_{12} = S_1, S_2, S_3$ and Σ_{14}, Σ_{24}, $\Sigma_{34} = T_1$, T_2, T_3, we have from (38)

$$\left.\begin{array}{lll}
[\beta_1, S_1] = 0, & [\beta_1, S_2] = i\beta_3, & [\beta_1, S_3] = -i\beta_2, \\
[\beta_2, S_1] = -i\beta_3, & [\beta_2, S_2] = 0, & [\beta_2, S_3] = i\beta_1, \\
[\beta_3, S_1] = i\beta_2, & [\beta_3, S_2] = -i\beta_1, & [\beta_3, S_3] = 0, \\
[\beta_4, S_1] = 0, & [\beta_4, S_2] = 0, & [\beta_4, S_3] = 0, \\
[\beta_1, T_1] = -i\beta_4, & [\beta_1, T_2] = 0, & [\beta_1, T_3] = 0, \\
[\beta_2, T_1] = 0, & [\beta_2, T_2] = -i\beta_4, & [\beta_2, T_3] = 0, \\
[\beta_3, T_1] = 0, & [\beta_3, T_2] = 0, & [\beta_3, T_3] = -i\beta_4, \\
[\beta_4, T_1] = i\beta_1, & [\beta_4, T_2] = i\beta_2, & [\beta_4, T_3] = i\beta_3,
\end{array}\right\} \qquad \ldots(39)$$

whence we have

$$[\beta_k, S_l]=i\beta_m, \ [S_k, \beta_l]=i\beta_m, \text{ for } k, l, m=\text{cycl. } (1,2,3), \qquad \text{...(40a)}$$

$$[\beta_4, T_k]=i\beta_k, \ [T_k, \beta_k]=i\beta_4, \ k=1,2,3. \qquad \text{...(41a)}$$

The solutions of (38) for $\Sigma_{\mu\nu}$ are evidently functions of the matrices β_μ. Following Bhabha (1945, 1949) the simplest assumption that can be made is that $\Sigma_{\mu\nu}$ contain no term that is a product of more than two β-matrices. Hence, since $\Sigma_{\mu\nu}$ is an antisymmetric tensor, it should have the form

$$i\Sigma_{\mu\nu}=q^2(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu) \qquad \text{...(42)}$$

where q^2 is a constant c-number. The commutation relations for β_μ can now be obtained directly using the relations (40a) and (41a) and following the method given in the Appendix. Rather than work out this in detail we adopt the following method.

(42) can also be written in terms of S_k and T_k as

$$q^2[\beta_k, \beta_l]=i\, S_m, \text{ for } k, l, m=\text{cycl. } (1,2,3,) \qquad \text{...(40b)}$$

$$q^2[\beta_k, \beta_4]=iT_k, \ k=1,2,3 \qquad \text{...(41b)}$$

The commutation relations (40a), (40b), 1(4a) and (14b) show that T_k and $q\beta_\mu$ will have the same eigen values as S_k (Hepner, 1951). Since for spin one the eigenvalues of S_k are -1, 0, 1, these are also the eigenvalues of the matrices $q\beta_\mu$ and T_k. Thus

$$q^2\beta_\mu{}^3=\beta_\mu, \ T_k{}^3 = T_k. \qquad \text{...(43)}$$

The second relation has already been obtained in (23). Since by (41b), T_k are expressed in terms of β_k, hence in order that the expression (32) for S_k may be consistent with the expression (40b), we should have

$$q^2=1. \qquad \text{...(44)}$$

Hence we obtain

$$\beta_\mu{}^3=\beta_\mu, \qquad \text{...(45)}$$

$$i\Sigma_{\mu\nu}=(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu) \qquad \text{...(46)}$$

By the substitution of (46) in (38), we get

$$\beta_\mu\beta_\lambda\beta_\nu-\beta_\mu\beta_\nu\beta_\lambda-\beta_\lambda\beta_\nu\beta_\mu+\beta_\nu\beta_\lambda\beta_\mu=\delta_{\mu\lambda}\beta_\nu-\delta_{\mu\nu}\beta_\lambda \qquad \text{...(47)}$$

For $\mu=\nu\neq\lambda$, in particular,

$$2\beta_\mu\beta_\lambda\beta_\mu-\beta_\mu^2\beta_\lambda-\beta_\lambda\beta_\mu^2=-\beta_\lambda, \ (\mu\neq\lambda), \qquad \text{...(48)}$$

Multiplying on both sides by β_μ and using (45), we have

$$2\beta_\mu^2\beta_\lambda\beta_\mu^2=\beta_\mu\beta_\lambda\beta_\mu, \ (\mu\neq\lambda)$$

Again multiplying on both sides by β_μ, we obtain

$$2\beta_\mu\beta_\lambda\beta_\mu=\beta_\mu^2\beta_\lambda\beta_\mu^2$$

Hence

$$\beta_\mu\beta_\lambda\beta_\mu=0, \qquad \text{...(49)}$$

Consequently (48) reduces to

$$\beta_\mu^2\beta_\lambda+\beta_\lambda\beta_\mu^2=\beta_\lambda, \ (\mu\neq\lambda), \qquad \text{...(50)}$$

Further for $\mu\neq\nu\neq\lambda$, (47) becomes

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu$$

Multiplying by β_ν^2 from the left and using (45) and (49)

$$\beta_\nu^2\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=0$$

We then obtain using (50) and (49)

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu=0, \ (\mu\neq\nu\neq\lambda) \qquad \text{.. (51)}$$

Hence $\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\lambda\beta_\mu$ which is symmetric in μ and ν should have the form

$$\beta_\mu\beta_\lambda\beta_\nu+\beta_\nu\beta_\mu\beta_\lambda=\beta_\mu\delta_{\lambda\nu}+\beta_\nu\delta_{\mu\lambda}, \qquad \text{. (35)}$$

which includes (45), (49), (50) and (51), These are the well-known Duffin-Kemmer commutation relations of β-matrices in the theory of spin one.

A four-vector β_μ, which is the Duffin-Kemmer representation, thus exists in spin-space, and has been obtained via Lorentz transformation from the non-relativistic representation of the spin. A scalar can then be formed by contracting β_μ with the four-vector $D_\mu=\frac{\partial}{\partial x_\mu}-\frac{ie}{\hbar c}A_\mu$, where

A_μ is the four-potential. This provides us the possibility of constructing a first order covariant wave equation for a particle of spin one.

$$(\beta_\mu D_\mu + \kappa\psi) = 0 \qquad ...(52)$$

where κ is to be so identified that one obtains in the field free case the Klein-Gordon equation as a second order equation. It is well known that on identification κ is found to be $mc/\hbar$.

Appendix

We have from (31)

$$S_i T_j - T_j S_i = i T_k,\ i, j, k = \text{cycl}\ (1, 2, 3). \qquad ...(A1)$$

By multiplication of (A1) on the left with S_i and on the right also with S_i and on subtraction

$$S_i^2\, T_j + T_j S_i^2 - 2\, S_i T_j S_i = i\,(S_i T_k - T_k S_i) = T_j \qquad ...(A2)$$

and by multiplication of this on both sides by S_i and with the use of $S_i^3 = S_i$

$$S_i T_j S_i = 2 S_i^2 T_j S_i$$

Again multiplying on both sides by S_i

$$S_i^2 T_j S_i^2 = 2 S_i T_j S_i^2$$

Therefore

$$S_i T_j S_i = 0,\ (i \neq j), \qquad ...(A3)$$

and, by (A2)

$$S_i^2 T_j + T_j S_i^2 = T_j,\ (i \neq j) \qquad ...(A4)$$

By multiplication of (A1) on the right with $T_k S_i$ and on the left with $S_i T_k$ and on addition

$$S_i(T_i T_k - T_k T_i) S_i = i(T_k^2 S_i + S_i T_k^2)$$

with the use of (A3). Therefore, since $S_i^3 = S_i$,

$$T_k^2 S_i + S_i T_k^2 = S_i,\ (i \neq k), \qquad ...(A5)$$

if

$$iS_i=(T_jT_k-T_kT_j),\quad i, j, k=\text{cycl}\,(1, 2, 3) \qquad \text{...(A6)}$$

When (A1) is multiplied on the left with T_j and on the right also with T_j and subtracted

$$2T_jS_iT_j-(T_j^2S_i+S_iT_j^2)=i(T_jT_k-T_kT_j)=-S_i$$

Therefore, by (A5)

$$T_jS_iT_j=0,\quad (i\neq j) \qquad \text{...(A7)}$$

Multiplying (A1) on both sides by T_j and using (A7)

$$T_jT_kT_j=0,\quad (i\neq j), \qquad \text{...(A8)}$$

By the substitution of (A6) in (A1) and with the use of (A8)

$$T_j^2T_k+T_kT_j^2=T_k,\ (j\neq k), \qquad \text{...(A9)}$$

By multiplication of (A1) on both sides by T_k

$$iT_k^3=T_kS_iT_jT_k-T_kT_jS_iT_k$$

Hence, since $S_iT_k-T_kS_i=-iT_j$, with the use of (A8) and (A9)

$$T_k^3=T_j^2T_k+T_kT_j^2=T_k \qquad \text{...(A10)}$$

Finally multiplying (A1) on the left and on the right with T_i and T_j respectively and reversing the order of multiplication, and on addition

$$T_iT_kT_j+T_jT_kT_i=0, \qquad \text{...(A11)}$$

where (A7), (A5), (A9) and $T_iS_i=S_iT_i$ have been used.

By substitution of S_i given by (A6) in (26a)—(27b), (28a′) and (28b′), we find that these are identically satisfied. Thus all the relations between T'-s and between T's and S's are direct consequence of (31). Further S_i in (A6) satisfy all the commutation relations (8a)—(8d) if the commutation relations (34) for T_k are utilized.

References

Bhabha, H. J. 1945 *Rev. Mod. Physics*, **17**, 300.

— — 1949 *Rev. Mod. Phys.*, **21**, 451.

Duffin, R. J. 1938 *Phys. Rev.*, **54**, 114.

Feshbach, H. & Villars, F. 1958 *Rev. Mod. Phys.*, **30**, 24.

Hefner, W. A. 1951 *Phys. Rev.*, **84**, 744.

Kemmer, N. 1939 *Proc. Roy. Soc.*A, **173**, 91.

Powell, J. L. & Crasemann, B. 1963 Quantum Machanics, 345, 369.